Solitary Front-like Wave Solutions of the Korteweg-deVries-Burgers' Equation

M. Lakshmanan and S. Ganesan

Department of Physics, Bharathidasan University, Tiruchirapalli, India

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Existence of a class of well-behaved solitary front-like wave solutions having arbitrary velocities for the Korteweg-deVries-Burgers' equation is pointed out. These solutions are analogous to the solitary wave solutions occurring in other well known nonlinear diffusive systems.

The Korteweg-de Vries equation perturbed by a Burgers' like dissipative term in the form

$$u_t - \mu u u_x + \varrho u_{xxx} = \nu u_{xx}$$

with

$$0 < \mu, \varrho, \nu \in R \tag{1}$$

is a physically important nonlinear Galilean-invariant system describing slow changes in linear waves due to a combination of nonlinearity, dispersion and dissipative effects [1]. When the dissipation is dominant, (1) is known to possess shock-like solitary waves [2] while in the pure dispersive limit ($\nu = 0$), (1) admits the well known K-dV solitons [3]. In the case of weak dissipation in (1), the existence of tailed quasi-soliton solutions has been investigated numerically and analytically by Fernandez et al. [4]. Using Lie's method of continuous transformation groups, Lakshmanan and Kaliappan [5] have deduced a class of invariant (similarity) solutions for (1), which are not in general of the Painlevé type. It is the aim of this note to point out the existence of a class of well-behaved solitary wave solutions of the front-type having arbitrary velocities for (1) for nonzero dissipation and dispersion, and then to compare its nature with the solitary fronts of certain nonlinear diffusive equations.

Following [5], it can be shown that (1) admits the three parameter (α, β, δ) invariant similarity variables

$$u = (\beta/\alpha) t + f(\zeta),$$
$$\zeta = x + (\mu \beta / 2 \alpha) t^2 - (\delta/\alpha) t. \tag{2}$$

Correspondingly, for the above similarity solutions, (1) can be reduced to the form

$$\alpha \varrho f''' - \alpha \nu f'' - (\alpha \mu f + \delta) f' + \beta = 0 \tag{3}$$

or

$$f'' - (\nu/\varrho) f' - (\mu / 2 \varrho) f^2 - (\delta / \alpha \varrho) f + (\beta / \alpha \varrho) \zeta + c_1 = 0, \tag{4}$$

where c_1 is an integration constant and the prime stands for differentiation with respect to ζ. Making the transformations

$$Z = (25 \mu \varrho / 12 \nu^2)^{\frac{1}{2}} \exp[(\nu / 5 \varrho) \zeta],$$
$$f = W(Z) \exp[(2 \nu / 5 \varrho) \zeta] - \frac{1}{\mu} \left(\frac{6 \nu^2}{25 \varrho} + \frac{\delta}{\alpha} \right), \tag{5}$$

(4) can be rewritten as

$$\mathrm{d}^2 W / \mathrm{d} Z^2 = 6 W^2 + [K_1 - K_2 \ln \{(12 \nu^2 / 25 \mu \varrho)^{\frac{1}{2}} Z\}] 1/Z^4, \tag{6}$$

with the constants

$$K_1 = \frac{625 \varrho^2}{12 \nu^4} \left[\frac{18 \nu^4}{625 \varrho^2} - c_1 \mu \varrho - \frac{\delta^2}{2 \alpha^2} \right],$$
$$K_2 = \frac{3625}{12} \cdot \frac{\beta \mu \varrho^3}{\alpha \nu^5}. \tag{7}$$

Equation (6) is in general not of the Painlevé type [6] and the solutions contain movable critical points except for a very special case discussed below.

While explicit solutions for the general case of (6) seem not possible, definite solutions may be given for the special case when $K_1 = 0$ and $K_2 = 0$. From (7), it is seen that this particular case corresponds to $\beta = 0$ and

$$(\delta/\alpha) = \frac{6 \nu^2}{25 \varrho} \left[1 - \frac{625 \mu \varrho^3 c_1}{18 \nu^4} \right]^{1/2}. \tag{8}$$

Correspondingly the invariant equation (4) takes the form

$$f'' - (\nu/\varrho) f' - (\mu / 2 \varrho) f^2 - (\delta / \alpha \varrho) f + c_1 = 0. \tag{9}$$

Reprint requests to Dr. M. Lakshmanan, Department of Physics, Bharathidasan University, Tiruchirapalli-620020, Indien.

Then corresponding to (9), (6) takes the form

$$\frac{d^2 W}{dZ^2} = 6\, W^2 \quad \text{or} \quad \left(\frac{dW}{dZ}\right)^2 = 4\, W^3 - c_2\,, \qquad (10)$$

where c_2 is the second integration constant. Equation (10) is a standard example to explain the important role played by integration constants in nonlinear differential equations [7]. Accordingly we may consider the two cases in which the arbitrary constant $c_2 \neq 0$ and $c_2 = 0$ separately.

If $c_2 \neq 0$, (10) may be explicitly integrated to give

$$W = \sqrt[3]{(c_2/4)} \cdot \left[\frac{(1+\sqrt{3}) - (1-\sqrt{3})\,\mathrm{cn}\,\{\sqrt[4]{3}\,(\sqrt[3]{4}\,\sqrt[6]{c_2}\,Z + \tilde{c}_3)\}}{1 - \mathrm{cn}\,\{\sqrt[4]{3}\,(\sqrt[3]{4}\,\sqrt[6]{c_2}\,Z + \tilde{c}_3)\}}\right], \qquad (11)$$

where $\tilde{c}_3$ is an integration constant and the square of the modulus of the Jacobian elliptic function $k^2 = (2-\sqrt{3})/4$. Using (11) and (5) we thus obtain a class of wave profiles which is singular, analogous to the one discussed recently by Kalinowski and Grundland [8].

Apart from the singular wave solutions discussed above, (1) possesses a class of solitary wave solutions. To obtain this, we take the case in which the integration constant c_2 in (10) is zero. Then (10) is trivially integrated to yield

$$W = 1/(Z + \tilde{p})^2\,, \qquad (12)$$

where $\tilde{p}$ is an integration constant. From this we obtain, after using the definition (5) and the condition (8), the solitary wave solution

$$u(x,t) = \frac{6\,\nu^2}{25\,\mu\,\varrho}\left[\frac{2}{\{1 + p\exp[(-\nu/5\,\varrho)\,(x - v\,t)]\}^2} - \left(1 + \frac{25\,\varrho}{6\,\nu^2}\,v\right)\right], \qquad (13)$$

where

$$p = \left(\frac{12\,\nu^2}{25\,\mu\,\varrho}\right)^{1/2} \tilde{p} \quad \text{and}$$

$$v = \frac{6\,\nu^2}{25\,\varrho}\left(1 - \frac{625\,\mu\,\varrho^3\,c_1}{18\,\nu^4}\right)^{1/2}.$$

For the choice $p > 0$, (13) corresponds to front-like solitary waves travelling with arbitrary velocity v, whose amplitude varies from

$$\frac{-6\,\nu^2}{25\,\mu\,\varrho}\left(1 + \frac{25\,\varrho}{6\,\nu^2}\,v\right) \text{ at } x = -\infty \text{ to}$$

$$\frac{-6\,\nu^2}{25\,\mu\,\varrho}\left(-1 + \frac{25\,\varrho}{6\,\nu^2}\,v\right) \text{ at } x = +\infty.$$

It is of interest to consider the limiting forms of (13) in the pure dispersive and dissipative cases. We note that (i) in the $\nu \to 0$, K-dV limit, $v \to (2\,\varrho\,c_1\,\mu)^{\frac{1}{2}}$, $c_1 < 0$ and so $u(x,t) \to$ constant. Thus the solitary wave (13) of the full KdVB equation is the extension of the constant solution of the KdV equation. Further (ii) in the $\varrho \to 0$ limit from (9) we find that $u(x,t) = -(v/\mu)\,(1 + \mathrm{K}\exp\theta)^{-1} \cdot 2\,K\exp\theta$, $\theta = -(v/\nu)\,(x - v\,t)$, $K = \text{constant} > 0$, the usual shock-like solitary wave solution of Burgers' equation, which is regular for all values of ν and has an infinite slope at $\theta = 0$ as $\nu \to 0$.

Finally we point out that the phenomenon reported in this paper is also analogous to the case of Fisher's equation $\Phi_t - \Phi_{xx} - \Phi + \Phi^n = 0$ for $n = 2$ and $n = 3$, where again singular elliptic function profiles and solitary waves arise. The important difference between the KdVB equation (1) and Fisher's equation is that the known explicit solitary wave solution for the latter case has a finite wave speed [9 – 11], while it is arbitrary in the KdVB case. In particular for $n = 2$ [9],

$$\Phi = [1 + K\exp\{(x - v\,t)/\sqrt{6}\}]^{-2}, \quad v = 5/\sqrt{6},$$

$K = \text{constant} > 0$ and for $n = 3$ (Nagumo's equation) [10], $\Phi = [1 + K\exp\{(x - v\,t)/\sqrt{2}\}]^{-1}$, $v = 3/\sqrt{2}$, $K > 0$. We may also point out that the Spalding equation, which describes the one dimensional laminar flame propagation [12]

$$\Phi_t - \mathrm{d}\Phi_{xx} - \hat{\beta}\,\Phi^{m+1}(\hat{\alpha} - \Phi^m) = 0$$

has got the solitary wave solution of the type $\Phi = [\hat{\alpha}/\{1 + \exp(m\,v\,(x - v\,t)/d)\}]^{1/m}$ again for the finite wave speed $v = \check{\alpha}(\hat{\beta}\,d/m + 1)^{1/2}$, where $\hat{\alpha}$, $\hat{\beta}$ and d are constants.

The KdVB equation has got travelling pulses (fronts) with arbitrary velocity essentially due to its third order nature, as there is an extra integration constant available in (9) compared to the solutions of the above nonlinear diffusion equations. Cornille and Gervois [13] have discussed the possibility of constructing bi-soliton solutions for some second order nonintegrable cases. It is interesting to investigate the existence of such bisoliton solutions for the present front-like solution and also to study the stability properties of the above solitary wave.

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